Probability theory - Class test 1

27 Feb, 2010, 10:00-12:00

Answer any five questions

Problem 1. [6 marks + 4 marks]

(1) Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R}^d)$. Assume that μ_n has density f_n and μ has density f w.r.t Lebesgue measure on $\mathbb{R}^d$. If $f_n(t) \to f(t)$ for a.e. t (with respect to Lebesgue measure), then show that $\mu_n \xrightarrow{d} \mu$.

(2) Show that $N(\mu_n, \sigma_n^2) \xrightarrow{d} N(\mu, \sigma)$ if and only if $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2$.

Problem 2. [6 marks + 4 marks]

(1) Let $X \sim \Gamma(\alpha, 1)$ and $Y \sim \Gamma(\alpha', 1)$ be independent random variables on a common probability space. Find the distribution of $\frac{X}{X+Y}$.

(2) With X and Y as above, find the distribution of $X+Y$.

Problem 3. [4 marks + 6 marks] On the probabiity space $([0,1], \mathcal{B}, \mathbf{m})$, for $k \geq 1$, define the functions

$$X_k(t) := \begin{cases} 0 & \text{if } t \in \bigcup_{j=0}^{2^{k-1}-1} [\frac{2j}{2^k}, \frac{2j+1}{2^k}). \\ 1 & \text{if } t \in \bigcup_{j=0}^{2^{k-1}-1} [\frac{2j+1}{2^k}, \frac{2j+2}{2^k}) \text{ or } t = 1. \end{cases}$$

(1) For any $n \geq 1$, what is the distribution of X_n?

(2) For any fixed $n \geq 1$, find the joint distribution of $(X_1, \ldots, X_n)$.

Problem 4. [6 marks + 4 marks]

(1) Let $\Omega = \{1, 2, \ldots, n\}$. For a probability measure $\mathbf{P}$ on Ω, we define it "entropy" $H(\mathbf{P}) := -\sum_{k=1}^n p_k \log p_k$ where $p_k = \mathbf{P}\{k\}$ and it is understood that $x \log x = 0$ if $x = 0$. Show that among all probability measures on Ω, the uniform probability measure (the one with $p_k = \frac{1}{n}$ for each k) is the unique maximizer of entropy.

(2) Let X be a non-negative random variable. If $\mathbf{E}[X] \leq 1$, then show that $\mathbf{E}[X^{-1}] \geq 1$.

Problem 5. [4 marks + 6 marks]

(1) If $\mu_n \ll \nu$ for each n and $\mu_n \xrightarrow{d} \mu$, then is it necessarily true that $\mu \ll \nu$? If $\mu_n \perp \nu$ for each n and $\mu_n \xrightarrow{d} \mu$, then is it necessarily true that $\mu \perp \nu$? In either case, justify or give a counterexample.

(2) Suppose X, Y are independent (real-valued) random variables with distribution μ and ν respectively. If at least one of μ or ν is absolutely continuous w.r.t Lebesgue measure, show that the distribution of $X+Y$ is also absolutely continuous w.r.t Lebesgue measure.

Problem 6. [5 marks + 5 marks]

(1) Suppose $\{\mu_\alpha : \alpha \in I\}$ and $\{\nu_\beta : \alpha \in J\}$ are two families of Borel probability measures on $\mathbb{R}$. If both these families are tight, show that the family $\{\mu_\alpha \otimes \nu_\beta : \alpha \in I, \beta \in J\}$ is also tight.

(2) Suppose $\{X_\alpha : \alpha \in I\}$ and $\{Y_\beta : \alpha \in J\}$ are two families of real-valued random variables on the same probability space. If the two families are tight (i.e., the family of their distributions are tight), then show that the family $\{X_\alpha + Y_\beta : \alpha \in I, \beta \in J\}$ is also tight.